Exercise 21

Use the Fourier transform to solve the boundary-value problem

$$u_{xx} + u_{yy} = -x\exp(-x^2), \quad -\infty < x < \infty,\ 0 < y < \infty,$$

$u(x, 0) = 0,$ for $-\infty < x < \infty$, u and its derivative vanish as $y \to \infty$. Show that

$$u(x, y) = \frac{1}{\sqrt{4\pi}} \int_0^\infty [1 - \exp(-ky)] \frac{\sin(kx)}{k} \exp\left(-\frac{k^2}{4}\right) dk.$$

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x, y)\} = U(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x, y)\, dx,$$

which means the partial derivatives of u with respect to x and y transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k, t)$$

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = \frac{d^n U}{dy^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{xx} + u_{yy}\} = \mathcal{F}\{-xe^{-x^2}\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-x)e^{-x^2} e^{-ikx}\, dx$$

Transform the derivatives with the relations above.

$$(ik)^2 U + \frac{d^2U}{dy^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-i)e^{-x^2} \frac{d}{dk} e^{-ikx}\, dx$$

Expand the coefficient of U.

$$-k^2 U + \frac{d^2U}{dy^2} = (-i)\frac{d}{dk} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx}\, dx \tag{1}$$

We just need to find the Fourier transform of e^{-x^2}.

$$-k^2 U + \frac{d^2U}{dy^2} = (-i)\frac{d}{dk}\mathcal{F}\{e^{-x^2}\}$$

Looking in a table of Fourier transforms, we see that the Fourier transform of a Gaussian function is a Gaussian function.

$$-k^2 U + \frac{d^2U}{dy^2} = (-i)\frac{d}{dk}\left(\frac{e^{-\frac{k^2}{4}}}{\sqrt{2}}\right)$$

Evaluating the derivative, we have

$$\frac{d^2U}{dy^2} - k^2U = \frac{ike^{-\frac{k^2}{4}}}{2\sqrt{2}}. \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$\begin{aligned}
u(x,0) = 0 \quad &\rightarrow \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{0\} \\
&\qquad\qquad U(k,0) = 0 \qquad (2) \\
\lim_{y\to\infty} u(x,y) = 0 \quad &\rightarrow \quad \mathcal{F}\left\{\lim_{y\to\infty} u(x,y)\right\} = \mathcal{F}\{0\} \\
&\qquad \lim_{y\to\infty} \mathcal{F}\{u(x,y)\} = 0 \\
&\qquad \lim_{y\to\infty} U(k,y) = 0. \qquad (3)
\end{aligned}$$

Equation (1) is a second-order inhomogeneous ODE, so its general solution is the sum of a complementary and particular solution.

$$U(k,y) = U_c + U_p$$

U_c is the solution to the associated homogeneous equation,

$$\frac{d^2U_c}{dy^2} - k^2U_c = 0,$$

which can be written in terms of exponentials.

$$U_c = C_1(k)e^{|k|y} + C_2(k)e^{-|k|y}$$

The inhomogeneous term is constant with respect to y, so U_p must be a constant as well.

$$\underbrace{\frac{d^2U_p}{dy^2}}_{=\,0} - k^2U_p = \frac{ike^{-\frac{k^2}{4}}}{2\sqrt{2}}$$

Solving for U_p gives

$$U_p = -\frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}.$$

Consequently, the general solution is

$$U(k,y) = C_1(k)e^{|k|y} + C_2(k)e^{-|k|y} - \frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}.$$

For equation (3) to be satisfied, we require that $C_1(k) = 0$. Use equation (2) now to determine $C_2(k)$.

$$U(k,0) = C_2(k) - \frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k} = 0 \quad \rightarrow \quad C_2(k) = \frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}$$

So we have after factoring

$$U(k,y) = -\frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}(1 - e^{-|k|y}).$$

Now that $U(k,y)$ is solved for, we change back to $u(x,y)$ by taking the inverse Fourier transform of it.

$$\mathcal{F}^{-1}\{U(k,y)\} = u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,y)e^{ikx}\,dk$$

Plug $U(k,y)$ into the definition.

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -\frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}(1 - e^{-|k|y})e^{ikx}\,dk$$

Use Euler's formula to write e^{ikx} in terms of sine and cosine.

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -\frac{ie^{-\frac{k^2}{4}}}{2\sqrt{2}k}(1 - e^{-|k|y})(\cos kx + i\sin kx)\,dk$$

Distribute $-i$ and bring $\sqrt{2}$ in front of the integral.

$$u(x,y) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} \frac{e^{-\frac{k^2}{4}}}{2k}(1 - e^{-|k|y})(-i\cos kx + \sin kx)\,dk$$

Since a real solution for the PDE is desired, take the real part of $u(x,y)$.

$$u(x,y) = \operatorname{Re}\left[\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} \frac{e^{-\frac{k^2}{4}}}{2k}(1 - e^{-|k|y})(-i\cos kx + \sin kx)\,dk\right]$$

Only the term with sine remains.

$$u(x,y) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} \frac{e^{-\frac{k^2}{4}}}{2k}(1 - e^{-|k|y})\sin kx\,dk$$

Because the integrand is even with respect to k, we can integrate from 0 to ∞, provided we double the integral. Also, k only goes through positive values now, so the absolute value sign can be dropped.

$$u(x,y) = \frac{2}{\sqrt{4\pi}}\int_{0}^{\infty} \frac{e^{-\frac{k^2}{4}}}{2k}(1 - e^{-ky})\sin kx\,dk$$

Therefore,

$$u(x,y) = \frac{1}{\sqrt{4\pi}}\int_{0}^{\infty} (1 - e^{-ky})\frac{\sin kx}{k}e^{-\frac{k^2}{4}}\,dk.$$